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Inverse approach to study the planar polynomial vector field with algebraic solutions

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Abstract
We construct a polynomial differential system that admits a given set of invariant algebraic curves. For such a system we solve the Darboux problem (the existence of the Darboux first integral), the Poincaré problem (the existence of an upper bound for the degree of invariant algebraic curve) and study Hilbert's 16th problem for algebraic limit cycles (the existence of an upper bound for the number of algebraic limit cycles).

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1. Introduction

By definition a real (complex) planar polynomial differential system or simply a polynomial system is a differential system of the form

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \tag{1.0}$$

where the dependent variables x and y are real (respectively complex), the independent variable (time t) is real and functions P and Q are polynomial in x and y with real (complex) coefficients.

We denote by $n = \max(\deg P(x, y), \deg Q(x, y))$ the degree of the polynomial system (1.0).

In this paper, we are mainly interested in the polynomial differential system which possesses a given set of invariant algebraic curves.

We denote by $R[x, y]$ the ring of real polynomial in x and y. An algebraic curve

$$g(x, y) = 0 \qquad g \in R[x, y]$$

is called an invariant algebraic curve of the system (1.0) if the following condition holds

$$P(x, y)\partial_x g(x, y) + Q(x, y)\partial_y g(x, y) = K(x, y)g(x, y)$$

for some real polynomial K of degree at most $n - 1$. The function g is called a partial integral.

In what follows the derivatives of g with respect to x and y are designated as $\partial_x g$, $\partial_y g$.

Many papers are dedicated to the study of polynomial planar vector fields with algebraic solutions (see for instance [Darb, Poin, Baut, Sver, Dol, Reyn, Jouan, Kooij1, Kooij2, Christ, Prelle, Qin, Valeeva]).

It is always helpful to look at this problem from another point of view. In this paper, we take an alternative viewpoint of starting with a given set of algebraic invariant curves and determining the form of the system which has such a set of invariant curves.

This point of view was firstly developed by Eruguin in the paper [Erug]. In this paper, the author stated and solved the problem of constructing a planar vector field for which the given curve is its invariant. He obtained the following result:

Proposition. *The most general planar vector field for which*

$$g(x, y) = 0$$

is its invariant curve is

$$\begin{cases} \dot{x} = \nu(x, y)\partial_y g + a(x, y) \\ \dot{y} = -\nu(x, y)\partial_x g + b(x, y) \end{cases}$$

where ν, a, b *are arbitrary functions:*

$$\begin{cases} a(x, y)\partial_x g + b(x, y)\partial_y g = \Phi(x, y) \\ \Phi(x, y)|_{g(x,y)=0} = 0. \end{cases}$$

These ideas were developed by Galiullin and his followers [Gal].

By applying the Eruguin–Galiullin ideas we state and study a new inverse problem in the theory of ordinary differential equations [Ram, Sad1, Sad2]. We analyse the problem of constructing a planar vector field with $S \geqslant 1$ invariant curves, in particular we study the case when the given curves are algebraic. These ideas have been systematically developed in [Sad2].

A similar problem was discussed in [Christ *et al*]. The authors study the polynomial differential system in the plane which has a given invariant set of invariant algebraic curves. They proved the following theorem.

Theorem 1 ([Christ *et al*])**.** *Let* $C_i = 0$ *for* $i = 1, 2, \ldots, p$ *be irreducible invariant algebraic curves in* C^2 *and set* $r = \sum_{j=1}^{p} \deg C_i$. *We assume that all* C_i *satisfy the following generic conditions:*

(i) There are no points at which C_i *and its first derivatives are all vanished.*
(ii) The highest order terms of C_i *have no repeated factors.*
(iii) If two curves intersect at a point in the finite plane, they are transversal at this point.
(iv) There are no more than two curves $C_i = 0$ *meeting at any point in the finite plane.*
(v) There are no two curves having a common factor in the highest order terms.

Then any polynomial vector field $\mathbf{X}$ *of degree m tangent to all* $C_i = 0$ *satisfies one of the following statements.*

(a) If $r < m + 1$ *then*

$$\mathbf{X} = \left(\prod_{i=1}^{p} C_i\right)\mathbf{Y} + \sum_{i=1}^{p} h_i \left(\prod_{j=1, j\neq i}^{p} C_i\right)\mathbf{X}_{C_i} \tag{1.1}$$

where $\mathbf{X}_{C_i} = (-C_{iy}, C_{ix})$ *is a Hamiltonian vector field,* h_i *are polynomials of degree no more than* $m - r + 1$, *and* $\mathbf{Y}$ *is a polynomial vector field of degree no more than* $m - r$.

(b) If $r = m + 1$ then

$$\mathbf{X} = \sum_{i=1}^{p} \alpha_i \left(\prod_{j=1, j\neq i}^{p} C_i \right) \mathbf{X}_{C_i} \tag{1.2}$$

with $\alpha_i \in C$.

(c) If $r > m + 1$ then

$$\mathbf{X} = \mathbf{0}.$$

The aim of this paper is to construct the most general (as possible) polynomial vector field with given invariant algebraic curves and to analyse for the corresponding system of differential equations the problem of the Darboux integrability, the Poincaré problem, Hilbert's 16th problem for algebraic limit cycles.

In what follows a system composed of several systems, or a system with the same additional conditions, given by the indexed formulae, will be presented as the union of the correspondent expressions; i.e. if these expressions are indexed as $(i, j), (k, l), \ldots, (m, n)$ then a new system which satisfies all these expressions will be designated as $(i, j) + (k, l) + \cdots + (m, n)$.

2. Polynomial planar vector field of degree *n* with $S \geqslant 2$ invariant algebraic curve

In this section, we construct a planar polynomial vector field from a given set of algebraic curves and establish the generality of the constructed differential system.

First of all we will introduce some notation and notions.

Let $\mathbf{v}$ be the vector field for which the algebraic curves

$$g_j(x, y) = 0 \qquad j = 1, 2, \ldots, S \quad S \geqslant 2$$

are its invariants, i.e.

$$\mathbf{v}(g_j(x, y)) = K_j(x, y) g_j(x, y) \qquad K_j \in R[x, y]. \tag{2.1}$$

Definition 2.1. *Given the set of the invariant functions of the vector field* $\mathbf{v}$,

$$g_1, g_2, \ldots, g_S \qquad S \geqslant 2$$

i.e. conditions (2.1) hold. If among these functions there exist at least two functions for which the algebraic curve

$$(A) \quad \{g_1, g_2\} \equiv \partial_x g_1 \partial_y g_2 - \partial_y g_1 \partial_x g_2 = 0 \qquad j = 1, 2, \ldots, S$$

does not contain the trajectories of the vector field $\mathbf{v}$, *then we will say that the set of given functions satisfies the condition (A).*

Proposition 2.1. *Suppose that $g_1, g_2, \ldots, g_S$, are functions which satisfy the condition (A). Then the most general planar vector field of degree n with these functions as partial integrals, is described by the system*

$$\begin{cases} \dot{x} = \mu_1(x, y) g_1(x, y)\{g_2, x\} + \mu_2(x, y) g_2(x, y)\{x, g_1\} \\ \dot{y} = \mu_1(x, y) g_1(x, y)\{g_2, y\} + \mu_2(x, y) g_2(x, y)\{y, g_1\} \end{cases} \tag{2.2}$$

with the following restrictions,

$$\mu_1 g_1 \{g_2, g_j\} + \mu_2 g_2 \{g_j, g_1\} + \mu_j g_j \{g_1, g_2\} = 0 \tag{2.3}$$

where μ_j, $j = 1, 2, \ldots, S$ *are arbitrary rational functions:*

$$\begin{cases} \dot{g}_j = \{g_1, g_2\}\mu_j g_j \\ \{g_1, g_2\}\mu_j \in R[x, y] \\ \deg(\{g_1, g_2\}\mu_j) \leqslant n - 1 \qquad j = 1, 2, \ldots, S. \end{cases} \tag{2.4}$$

In fact, the vector field

$$v = \mu_1 g_1\{, g_2\} - \mu_2 g_2\{, g_1\} \tag{2.5}$$

is generated by the system (2.2) and from (2.4) it follows that

$$v(g_j) = \{g_1, g_2\}\mu_j g_j \qquad j = 1, 2.$$

Hence, by considering (2.3), we deduce that

$$v(g_j) = \mu_1 g_1\{g_j, g_2\} - \mu_2 g_2\{g_j, g_1\} = \mu_j g_j\{g_1, g_2\} \qquad j = 2, 3, \ldots, S$$

so the given functions are partial integrals of the vector field v.

To establish the fact that the constructed vector field v, under the condition (A) is the most general we shall suppose that

$$w = X(x, y)\partial_x + Y(x, y)\partial_y$$

is another vector field which admits the given partial integrals, i.e.,

$$X(x, y)\{y, g_j\} - Y(x, y)\{x, g_j\}|_{g_j=0} = 0 \qquad j = 1, 2, \ldots, S.$$

By choosing the arbitrary functions μ_1, μ_2 as follows,

$$\mu_j g_j = \frac{X(x, y)\{y, g_j\} - Y(x, y)\{x, g_j\}}{\{g_1, g_2\}} \qquad j = 1, 2$$

and by inserting them into (2.5) we easily deduced that under the condition (A) the vector field v coincides with w.

The following examples demonstrate the necessity of the condition (A) for the system (2.2) + (2.3).

Example 2.1. The quadratic system

$$\begin{cases} \dot{x} = y(ax - b(y - 1)) + x^2 + y^2 - 1 \\ \dot{y} = -x(ax - b(y - 1)) + a(x^2 + y^2 - 1) \end{cases}$$

which admits the invariant curves

$$y - 1 = 0 \qquad x^2 + y^2 - 1 = 0$$

cannot be obtained from (2.2). The function

$$\{x^2 + y^2 - 1, y - 1\} = 2x$$

vanishes at the point (0, 1) which is the trajectory of the quadratic vector field.

Example 2.2. When the given algebraic curves are

$$g_1(x, y) = xy - 1 = 0 \qquad g_2(x, y) = y + 1 = 0 \qquad g_3(x, y) = y = 0$$

then it is evident that condition (A) does not hold because

$$\{xy - 1, y + 1\} = y.$$

The system (2.2) + (2.3) takes the following form:

$$\begin{cases} \dot{x} = \mu_2(x, y)(y + 1)x + \mu_1(x, y)(xy - 1) \\ \dot{y} = -\mu_2(x, y)(y + 1)y \\ \mu_2(x, y)(y + 1) = -\mu_3(x, y)y. \end{cases}$$

It is clear that this differential system has the given invariant curves for any polynomial μ_2. This fact is not compatible with the last equation.

In [Christ *et al*] the following result was proposed:

"Let $g_j(x, y) = 0$ for $j = 1, 2, \ldots, S$ be different irreducible invariant algebraic curves of system (1.0) with $\deg g_j(x, y) = c_j$. Assume that g_j satisfy the conditions (i), (iii) and (iv) of theorem 1.

Then

(a) if $(\partial_x g_j, \partial_y g_j) = 1$ for $j = 1, 2, \ldots, S$ then system (1.0) has the normal form

$$\begin{cases} \dot{x} = \left(A_0 - \displaystyle\sum_{j=1}^{S} \frac{A_j \partial_y g_j(x, y)}{g_j(x, y)}\right) \displaystyle\prod_{k=1}^{S} g_j(x, y) \\ \dot{y} = \left(B + \displaystyle\sum_{j=1}^{S} \frac{A_j \partial_x g_j(x, y)}{g_j(x, y)}\right) \displaystyle\prod_{k=1}^{S} g_j(x, y) \end{cases} \tag{2.6}$$

where B, A_0 and A_j are suitable polynomials ...".

The relation between the model proposed in [Christ *et al*] and the model proposed by us is established in the following consequence.

Corollary 2.1. *The system*

$$\begin{cases} \dot{x} = \displaystyle\sum_{j=1}^{S} h_j(x, y) \prod_{k\neq j}^{S} g_k \{g_j, x\} + \frac{h_{S+2}}{\{g_1, g_2\}} \prod_{k=1}^{S} g_k \equiv P(x, y) \\ \dot{y} = \displaystyle\sum_{j=1}^{S} h_j(x, y) \prod_{k\neq j}^{S} g_k \{g_j, y\} - \frac{h_{S+1}}{\{g_1, g_2\}} \prod_{k=1}^{S} g_k \equiv Q(x, y) \end{cases} \tag{2.7}$$

is a particular case of the system (2.2) + (2.3), where $g_{S+1} = x$, $g_{S+2} = y$,$h_1, \ldots, h_{S+2}$ *are arbitrary polynomials such that*

$$n = \max(\deg P, \deg Q).$$

In fact, by considering the identity

$$\{g_k, g_j\}\{g_i, g_m\} + \{g_k, g_m\}\{g_j, g_i\} + \{g_k, g_i\}\{g_m, g_j\} = 0$$

we see that the polynomials

$$g_j \mu_j(x, y)\{g_1, g_2\} = \sum_{m=1}^{S+2} h_m \{g_m, g_j\} \prod_{k\neq m}^{S} g_k \qquad j = 1, 2, \ldots, S$$

are solutions of (2.3). By inserting these polynomials into (2.2) we obtain the system (2.7).

Corollary 2.2. *The system (2.7) coincides with the system (2.6). In fact, if we introduce the following denotation,*

$$h_j = A_j \qquad j = 1, 2, \ldots, S \qquad h_{s+1} = -\{g_1, g_2\} B, h_{S+2} = \{g_1, g_2\} A_0$$

we obtain the differential system (2.6).

The following two examples show that the system (2.2) + (2.5) is more general than the model proposed in [Christ *et al*].

Example 2.3. The system

$$\begin{cases} \dot{x} = 3\mu_3(x, y)(y-1) - \mu_2(x, y)(4x+3y+5) \\ \dot{y} = -4\mu_3(x, y)(y-1) \end{cases}$$

has as particular integrals the functions

$$g_1(x, y) = x^2 + y^2 - 1 \qquad g_2(x, y) = 4x + 3y + 5 \qquad g_3(x, y) = y - 1$$

iff the following condition holds

$$4\mu_1 g_1 - 2x\mu_2 g_2 + 2(3x - 4y)\mu_3 g_3 = 0. \tag{2.8}$$

By considering that

$$\{4x + 3y + 5, y - 1\} = 4$$

we can see that the condition (A) in this case holds.

After some calculations we can prove that

$$\begin{cases} \mu_1 = 2(x + 2y) \\ \mu_2 = 2y + x - 2 \\ \mu_3 = x + 2y + 2 \end{cases} \tag{2.9}$$

satisfies (2.8). For this case the constructed differential system is transformed into the quadratic system,

$$\begin{cases} \dot{x} = (2x - y + 1)y + x^2 + y^2 - 1 \\ \dot{y} = -(2x - y + 1)x + 2(x^2 + y^2 - 1) \end{cases} \tag{2.10}$$

which cannot be obtained from (2.6) [Christ *et al*] .

Example 2.4. The system

$$\begin{cases} \dot{x} = \mu_1(x, y)x \\ \dot{y} = \mu_2(x, y)y \\ \mu_1(x, y)x + \mu_2(x, y)y - \mu_3(x, y)(x + y) = 0 \end{cases}$$

has as partial integrals the functions

$$\begin{cases} g_1 = x \\ g_2 = y \\ g_3 = x + y \end{cases}$$

which satisfy the condition (A).

Evidently,

$$\begin{cases} \mu_1 = b(x, y)y + a(x, y) \\ \mu_2 = -b(x, y)x + a(x, y) \\ \mu_3 = a(x, y) \end{cases}$$

are solutions of the above equation, where a and b are suitable polynomials of degree $n - 2$ and $n - 1$ respectively.

Hence we obtain the polynomial system of degree n:

$$\begin{cases} \dot{x} = x(b(x, y)y + a(x, y)) \\ \dot{y} = y(-b(x, y)x + a(x, y)). \end{cases}$$

In the case when

$$a(x, y) = 1 + x + x^2 + xy + y^2 \qquad b(x, y) = 1 - y$$

we obtain the cubic system [Christ *et al*]

$$\begin{cases} \dot{x} = x(1 + x + x^2 + y + yx) \\ \dot{y} = y(1 + x^2 + 2xy + y^2). \end{cases}$$

In all statements below we restrict ourselves only to the particular case when the functions μ_j, $j = 1, 2, \ldots, S$, are

$$\begin{cases} \mu_j \in R[x, y] \\ \deg(\mu_j) \leqslant n - 1 - r \qquad r = \deg(\{g_1, g_2\}). \end{cases} \tag{2.11}$$

Now we will study the Darboux integrability and the Poincaré problem for the system (2.2) + (2.3) + (2.11).

3. Darboux integrability and the Poincaré problem for the differential system (2.2) + (2.3) + (2.11)

In this section, we solve the problem of the Darboux integrability and the Poincaré problem for the system constructed above.

3.1. Darboux integrability

Proposition 3.1. *Let us consider the polynomial system (2.2) + (2.3) + (2.11) of degree n. Let us suppose that this system admits S algebraic particular integrals $g_j(x, y)$ with $j = 1, 2, \ldots, S$, which are irreducible. Let*

$$\dim_R R_{n-r-1}[x, y] \qquad r = \deg\{g_1, g_2\}$$

be the dimension of the real space

$$R_{n-r-1}[x, y]$$

of polynomials of degree at most $n - r - 1$ in the independent variables x and y and let us suppose that

$$S > \dim_R R_{n-r-1}[x, y] = \tfrac{1}{2}(n - r)(n + 1 - r).$$

Then the function

$$F(x, y) = \prod_{j=1}^{S} |g_j|^{\lambda_j}$$

is the first integral of this system (Darboux first integral), where $\lambda_1, \lambda_2, \ldots, \lambda_S$ are constants and at least one of them is nonzero.

The proof is similar to the proof of the classical Darboux theorem [Darb, Christ, Chav].

The differential system, which admits a Darboux first integral is called Darboux integrable.

From proposition 3.1 we obtain the following consequences.

Corollary 3.1. *Suppose that in (2.2) + (2.3) + (2.11) the number $r = n - 1$, then the constructed polynomial vector field is Darboux integrable.*

In fact, the function

$$F(x, y) = \frac{|g_1|^{\lambda_1}}{|g_2|^{\lambda_2}}$$

where $\mu_1 = \mu_1^0, \mu_2 = \mu_2^0$ *are constants,* $\mu_1^0\lambda_1 + \mu_2^0\lambda_2 = 0$ *is Darboux first integral.*

Proposition 3.2. *The system (2.2) + (2.3) is Darboux integrable iff* μ_1, μ_2 *are such that*

$$g_m\mu_m(x, y) = \nu(x, y)\sum_{j=1}^{S}\lambda_j\{g_j, g_m\}\prod_{k\neq j}^{S} g_k \qquad m = 1, 2 \quad S \geqslant 2 \tag{3.1}$$

where λ_j, $j = 1, 2, \ldots, S$ *are constant,* ν *is an arbitrary rational function:*

$$0 \leqslant \deg(\{g_1, g_2\}\mu_m) \leqslant n - 1 \qquad m = 1, 2.$$

The proof is the following.

Suppose that F is the first integral of (2.2) + (2.3). Then the polynomials $\mu_1, \ldots, \mu_S$ satisfy the relations

$$\sum_{j=1}^{S}\lambda_j\mu_j = 0.$$

Therefore we obtain the following equalities if $S > 2$:

$$\begin{aligned}
\sum_{j=1}^{S}\lambda_j\mu_j &= \lambda_1\mu_1 + \lambda_2\mu_2 + \sum_{j=3}^{S}\lambda_j\left(\mu_1 g_1\frac{\{g_j, g_2\}}{g_j\{g_1, g_2\}} + \mu_2 g_2\frac{\{g_1, g_j\}}{g_j\{g_1, g_2\}}\right)\\
&= \frac{\mu_1 g_1}{\{g_1, g_2\}}\left(\lambda_1\frac{\{g_1, g_2\}}{g_1} + \sum_{j=3}^{S}\lambda_j\frac{\{g_j, g_2\}}{g_j}\right)\\
&\quad - \frac{\mu_2 g_2}{\{g_1, g_2\}}\left(\lambda_2\frac{\{g_2, g_1\}}{g_2} + \sum_{j=3}^{S}\lambda_j\frac{\{g_j, g_1\}}{g_j}\right)\\
&= \frac{\mu_1 g_1\left(\sum_{j=1}^{S}\lambda_j\{g_j, g_2\}\prod_{\substack{k=1\\k\neq j}}^{S} g_k\right) - \mu_2 g_2\left(\sum_{j=1}^{S}\lambda_j\{g_j, g_1\}\prod_{\substack{k=1\\k\neq j}}^{S} g_k\right)}{\{g_1, g_2\}\prod_{k=1}^{S} g_k}\\
&= 0.
\end{aligned}$$

From here we can trivially deduce the required representation for μ_j, $j = 1, 2$.

In the case when $S = 2$ we have

$$\sum_{j=1}^{2}\lambda_j\mu_j = 0$$

hence

$$\mu_1 = a(x, y)\lambda_2 \qquad \mu_2 = -a(x, y)\lambda_1$$

where a is an arbitrary rational function. In particular, if

$$a(x, y) = \nu(x, y)\{g_1, g_2\}$$

then we obtain formula (3.1).

The reciprocity can be obtained as follows.

Let μ_1, μ_2 be polynomials determined by (3.1). We will prove that in this case the system (2.2) + (2.3) admits Darboux first integral F.

In fact, by inserting μ_j, $j = 1, 2$ in (3.2) we get the equations

$$\begin{cases} \dot{x} = \nu(x, y) \displaystyle\sum_{j=1}^{S} \lambda_j \prod_{\substack{k=1\\k\neq j}}^{S} g_k(\{g_j, g_1\}\{x, g_2\} + \{g_j, g_2\}\{g_1, x\}) \\ \dot{y} = \nu(x, y) \displaystyle\sum_{j=1}^{S} \lambda_j \prod_{\substack{k=1\\k\neq j}}^{S} g_k(\{g_j, g_1\}\{y, g_2\} + \{g_j, g_2\}\{g_1, y\}). \end{cases}$$

By taking into consideration the identity

$$\{g_k, g_j\}\{g_i, g_m\} + \{g_k, g_m\}\{g_j, g_i\} + \{g_k, g_i\}\{g_m, g_j\} = 0$$

we obtain

$$\begin{aligned} &\{g_j, g_1\}\{x, g_2\} + \{g_j, g_2\}\{g_1, x\} = \{g_1, g_2\}\{g_j, x\} \\ &\{g_j, g_1\}\{y, g_2\} + \{g_j, g_2\}\{g_1, y\} = \{g_1, g_2\}\{g_j, y\}. \end{aligned}$$

Hence, the above equations take the form

$$\begin{cases} \dot{x} = \nu(x, y)\{g_1, g_2\} \displaystyle\sum_{j=1}^{S} \lambda_j \{g_j, x\} \prod_{\substack{k=1\\k\neq j}}^{S} g_k \\ \dot{y} = \nu(x, y)\{g_1, g_2\} \displaystyle\sum_{j=1}^{S} \lambda_j \{g_j, y\} \prod_{\substack{k=1\\k\neq j}}^{S} g_k \end{cases}$$

or equivalently,

$$\begin{cases} \dot{x} = \nu(x, y)\{g_1, g_2\} \displaystyle\prod_{k=1}^{S} g_k \sum_{j=1}^{S} \lambda_j \frac{\{g_j, x\}}{g_j} \\ \quad = -\nu(x, y)\{g_1, g_2\} \displaystyle\prod_{k=1}^{S} g_k \sum_{j=1}^{S} \lambda_j \frac{\partial_y g_j}{g_j} \\ \dot{y} = \nu(x, y)\{g_1, g_2\} \displaystyle\prod_{k=1}^{S} g_k \sum_{j=1}^{S} \lambda_j \frac{\{g_j, y\}}{g_j} \\ \quad = \nu(x, y)\{g_1, g_2\} \displaystyle\prod_{k=1}^{S} g_k \sum_{j=1}^{S} \lambda_j \frac{\partial_x g_j}{g_j}. \end{cases}$$

By considering the following expressions,

$$\begin{cases} \displaystyle\sum_{j=1}^{S} \lambda_j \frac{\partial_y g_j}{g_j} = \partial_y \ln \prod_{j=1}^{S} |g_j|^{\lambda_j} \\ \displaystyle\sum_{j=1}^{S} \lambda_j \frac{\partial_x g_j}{g_j} = \partial_x \ln \prod_{j=1}^{S} |g_j|^{\lambda_j} \end{cases}$$

and introducing the functions

$$\begin{cases} F(x, y) = \prod_{j=1}^{S} |g_j|^{\lambda_j} \\ V(x, y) = -\nu(x, y)\{g_1, g_2\} \prod_{k=1}^{S} g_k \end{cases}$$

finally we obtain the system

$$\begin{cases} \dot{x} = V(x, y)\partial_y \ln F \\ \dot{y} = -V(x, y)\partial_x \ln F \end{cases}$$

so the function F is a first integral of (2.2).

It is easy to show that the function V satisfies the equation

$$\dot{V} = \{V, \ln F\}V.$$

From this relation and the result given in [Gorb, Giancomini *et al*] we can deduce the condition for the existence of the periodical solution for the Darboux integrable differential system (2.2) + (2.3) + (2.11).

Example 3.1. The quadratic system (2.10) is Darboux integrable. In fact, in this case we have, from (3.1), that

$$\mu_1 = -2(y - 2x - 2)\nu(x, y)(2x + 4y)$$
$$\mu_2 = -2(y - 2x - 2)\nu(x, y)(x + 2y - 2)$$
$$\mu_3 = -2(y - 2x - 2)\nu(x, y)(x + 2y + 2)$$

if

$$-\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Hence we obtain that

$$\nu(x, y) = \frac{1}{4x - 2y + 4}.$$

The function

$$F(x, y) = \frac{(y - 1)(4x + 3y + 5)}{(x^2 + y^2 - 1)}$$

is a Darboux's first integral of the system (2.10).

Example 3.2. We will construct the polynomial vector field of degree 5 with the following set of partial integrals:

$$\begin{cases} g_1 \equiv x - 1 \quad g_2 \equiv x + 1 \quad g_3 \equiv x - (\sqrt{5} - 2) \quad g_4 \equiv x + (\sqrt{5} - 2) \\ g_5 \equiv y - 1 \quad g_6 \equiv y + 1 \quad g_7 \equiv y - (\sqrt{5} - 2) \quad g_8 \equiv y + (\sqrt{5} - 2). \end{cases}$$

This set of functions satisfies the condition (A), because

$$\{x - 1, y - 1\} = 1.$$

The functions $\mu_1, \mu_2 \equiv \mu_5$ are polynomials of the maximal degree 4 and are such that

$$\begin{cases} \mu_1 = (ax + by + c)(x + 1)(x^2 - (\sqrt{5} - 2)^2) \\ \mu_5 = (Ax + By + C)(y + 1)(y^2 - (\sqrt{5} - 2)^2). \end{cases}$$

The differential equations (3.2) take the form

$$\begin{cases} \dot{x} = (ax + by + c)(x^2 - 1)(x^2 - (\sqrt{5} - 2)^2) \\ \dot{y} = (Ax + By + C)(y^2 - 1)(y^2 - (\sqrt{5} - 2)^2). \end{cases}$$

From proposition 3.2 we deduce that the system is Darboux integrable iff

$$\begin{cases} \mu_1(x, y) = \nu(x, y) \prod_{k \neq 1}^{8} (y - a_j)(x - a_j) \sum_{j=5}^{8} \frac{\lambda_j}{y - a_j} \\ \mu_2(x, y) = \nu(x, y) \prod_{k \neq 2}^{8} (y - a_j)(x - a - j) \sum_{j=1}^{4} \frac{\lambda_j}{x - a_j} \\ a_1 = 1 \quad a_2 = -1 \quad a_3 = 2 - \sqrt{5} \quad a_4 = \sqrt{5} - 2. \end{cases}$$

By comparing the above system with (3.1), we obtain after some calculations

$$\begin{cases} \nu(x, y) \sum_{j=5}^{8} \frac{\lambda_j}{y - a_j} = \frac{ax + by + c}{\prod_{k=5}^{8} (y - a_j)} \\ \nu(x, y) \sum_{j=1}^{4} \frac{\lambda_j}{x - a_j} = \frac{Ax + By + C}{\prod_{k=1}^{4} (x - a_j)}. \end{cases}$$

Hence, $a = B = 0$, $\nu(x, y) = 1$ and

$$\lambda_j = \frac{Aa_j + C}{\prod_{k \neq j} (a_k - a_j)}$$

$$\lambda_{j+4} = \frac{ba_j + c}{\prod_{k \neq j} (a_k - a_j)} \qquad j = 1, 2, 3, 4.$$

The Darboux first integral is the following:

$$F(x, y) = \prod_{j=1}^{4} |y - a_j|^{Aa_j + C} |x - a_j|^{ba_j + c}.$$

It is evident that if $aB \neq 0$ then the constructed differential system is not Darboux integrable.

In particular, if we select

$$\begin{cases} a = 1 \quad b = \sqrt{5} \quad c = 0 \\ A = \sqrt{5} \quad B = 1 \quad C = 0 \end{cases}$$

then we obtain the system given in [Art *et al*]:

$$\begin{cases} \dot{x} = (x + \sqrt{5}y)(x^2 - 1)(x^2 - (\sqrt{5} - 2)^2) \\ \dot{y} = (\sqrt{5}x + y)(y^2 - 1)(y^2 - (\sqrt{5} - 2)^2). \end{cases}$$

This system is not Darboux integrable and has 14 invariant straight lines

$$g_9 \equiv y + x = 0 \qquad g_{11} \equiv y + kx + (1 - k) = 0 \qquad g_{13} \equiv y + \frac{x}{k} + \frac{1 - k}{k} = 0$$

$$g_{10} \equiv y - x = 0 \qquad g_{12} \equiv y + kx - (1 - k) = 0 \qquad g_{14} \equiv y + \frac{x}{k} - \frac{1 - k}{k} = 0$$

where $k = \frac{(\sqrt{5} - 1)}{2}$.

3.2. The Poincaré problem

We proceed with the study of the Poincaré problem for the system (2.2) + (2.3) + (2.11), of the existence of the upper bound of the degrees of the invariant algebraic curves [Carn].

If we suppose that the system (2.2) + (2.3) is Darboux integrable, then, in general, there does not exist an upper bound of the degrees of invariant algebraic curves.

In fact, the quadratic system

$$\begin{cases} \dot{x} = x(p_0(m-1)y + p_1 m x + p_2 m) \\ \dot{y} = y(p_0 k y + p_1(k+1)x + p_2 k) \end{cases}$$

admits the invariant curve

$$y^m + p_0 y x^k + p_1 x^{k+1} + p_2 x^k = 0$$

of arbitrary degree $l = \max(m, k+1)$.

It is easy to show that

$$F(x, y) = \frac{y^m + p_0 y x^k + p_1 x^{k+1} + p_2 x^k}{y^m}$$

is a Darboux first integral [Sad1].

Proposition 3.3. *Suppose the system (2.2) + (2.3) + (2.11) is not Darboux integrable. Then*

$$1 \leqslant \deg(g_j) \leqslant n \qquad j = 1, 2 \ldots, S \quad S \geqslant 2.$$

In fact, let us assume that the given polynomials g_1, g_2 are of the degree m and k, respectively, and admit the representations

$$\begin{cases} g_1(x, y) = x^{m_1} y^{m_2} + \cdots & m_1 + m_2 = m \\ g_2(x, y) = x^{k_1} y^{k_2} + \cdots & k_1 + k_2 = k. \end{cases}$$

By inserting these expressions into equations (3.2) we obtain

$$\begin{cases} \dot{x} = (\mu_2 m_2 - \mu_1 k_2) x^{m_1+k_1} y^{m_2+k_2-1} + \cdots = P(x, y) \\ \dot{y} = (\mu_1 k_1 - \mu_2 m_1) x^{m_1+k_1-1} y^{m_2+k_2} + \cdots = Q(x, y) \end{cases}$$

therefore we deduce that

$$m_1 + k_1 - 1 + m_2 + k_2 = m + k - 1 \leqslant n$$

and hence $\deg(g_j) \leqslant n$.

Of course, if

$$\begin{cases} \mu_2 m_2 - \mu_1 k_2 = 0 \\ \mu_1 k_1 - \mu_2 m_1 = 0 \end{cases}$$

then the system is Darboux integrable.

Example 3.3. Let us consider the following functions,

$$g_1(x, y) = x - a \qquad g_2(x, y) = \epsilon x^n + G(x, y)$$

where G is a polynomial of degree $n-1$:

$$\begin{cases} \{g_1, g_2\} = \partial_y G(x, y) \\ \deg \partial_y G(x, y) = n - 2. \end{cases}$$

The polynomial vector field of degree n

$$\begin{cases} \dot{x} = (Ax + By + C)(x - a)\partial_y g_2(x, y) \\ \dot{y} = -(Ax + By + C)(x - a)\partial_x g_2(x, y) + (\beta + n(Ax + By + C))g_2(x, y) \end{cases} \tag{3.2}$$

is not Darboux integrable iff $B\beta \neq 0$, where A, B, C, β, a are real constants such that

$$A(C + Aa) \neq 0.$$

In fact if $B = 0$ then the system (3.2) admits three partial algebraic integrals, after some calculations we can prove that

$$F(x, y) = |\epsilon x^n + G(x, y)|^{C+Aa}|x - a|^{\beta - n(Aa+C)}|Ax + C|^{-A\beta}$$

is the first integral of (3.2).

Let us suppose that (3.2) admits only two partial algebraic integrals g_1, g_2 and is Darboux integrable, i.e., there are numbers λ_1, λ_2 such that

$$\lambda_1\mu_1 + \lambda_2\mu_2 = 0$$

then, as a consequence we obtain

$$\mu_1 = -a(x, y)\lambda_2 \qquad \mu_2 = a(x, y)\lambda_1.$$

On the other hand,

$$\mu_1 = (Ax + By + C)\partial_y g \qquad \mu_2 = (\beta + n(Ax + By + C))\partial_y g$$

hence if $\beta = 0$ then

$$a(x, y) = (Ax + By + C)\partial_y g \qquad \lambda_1 = -n \qquad \lambda_2 = 1$$

and, as a consequence,

$$F(x, y) = \frac{(\epsilon x^n + G(x, y))}{(x - a)^n}$$

is a Darboux first integral.

4. The Hilbert 16th problem for algebraic limit cycles

In this section, we determine an upper and a lower bound for number of algebraic limit cycles for the non-Darboux integrable differential system (2.2) + (2.3) + (2.11).

Hilbert's 16th problem requires that the following problems are studied [Hil]:

(I) The ovals of the algebraic curves.
(II) The limit cycles of the polynomial planar vector field (to establish an upper bound for the number of limit cycles).

The part II of Hilbert 16th problem is now a days formulated as follows [Smal]: 'Consider the differential equation in R^2

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$$

where P and Q are polynomials. Is there a bound K of the number of limit cycles of the form

$$K \leqslant n^q$$

where n is the maximum of degrees of P and Q and q is a universal constant?'

There are also studies concerning an intermediate problem related to the upper bound of the number of limit cycles which can be generated from the Hamiltonian polynomial vector field after small polynomial perturbations. We will call this problem III.

The progress made towards solving problem II can be summarized by the following results:

Ilyashenko and Ecalle provided a correct proof of Dulac's theorem: "Every polynomial vector field has a finite number of limit cycles."

Nothing is known about the existence of an upper bound even for quadratic vector fields.

As far as problem III is concerned, the number of limit cycles of the polynomial planar vector field is closely related to the number of isolated zeros of the Abelian integral (the weakened Hilbert 16th problem.)

Varchenko and Khovansky gave the second important result. They proved that there is a number $\mathcal{N}(n)$ such that the number of zeros of the Abelian integral does not exceed $\mathcal{N}(n)$.

But even when $n = 2$ no explicit bound for $\mathcal{N}(n)$ is known.

The aim of this section is to study problem II when the limit cycles are algebraic [Baut, Qin].

As we observe in the introduction, there are a considerable number of studies which discuss the problem related to the algebraic solutions of the polynomial vector field (see, for instance, [Darb, Poin, Baut, Dol, Jouan, Koij1, Koij2, Christ *et al*]).

To solve the 16th Hilbert problem for algebraic limit cycles for the polynomial planar vector field it is necessary

(i) to determine the upper bound for the degree of the invariant irreducible algebraic curve $g(x, y) = 0$ (the Poincaré problem),

(ii) to construct the polynomial planar vector field with a fixed degree for which the given algebraic curve of the maximal degree with the maximal number of ovals is its invariant.

Of course, if the invariant algebraic curve of the polynomial system has m ovals then the system has usually m limit cycles [Sver]. Harnack's theorem claims that the maximum number of ovals of real algebraic curves of degree m is $\frac{1}{2}(m-1)(m-2)+1$. So we obtain that the maximal number of algebraic limit cycles which the polynomial planar vector field can admit is equal to or greater than $\frac{1}{2}(m-1)(m-2)+1$, where m is the degree of the invariant curve.

In the paper [Shi] the author affirms "Another artificial example is constructed as follows. We consider an algebraic curve $g(x, y) = 0$, where g is a polynomial of degree n in x, y. Suppose that this curve has M ovals. Then, the system

$$\begin{cases} \dot{x} = \partial_y g(x, y) \\ \dot{y} = -\partial_x g(x, y) + g(x, y) \end{cases}$$

has usually M algebraic limit cycles. For detail and more precision, see Sverdlov's paper."

It is evident that this system admits the following first integral which is analytic in the plane:

$$F(x, y) = g(x, y)\,\mathrm{e}^{-x}.$$

Let us denote by

$$A(n, S)$$

the maximal number of the algebraic limit cycles that the polynomial system (2.2) + (2.3) + (2.11) of degree n with S invariant irreducible algebraic curves can admit.

The aim of the assertions below is to determine a lower and an upper bound (when $S > 2$) for $A(n, S)$.

4.1. A lower bound for $A(n, 1)$

Let V be a function

$$V(x, y) = \int \prod_{j=1}^{n-1}(y - b_j)\,\mathrm{d}y + \prod_{j=1}^{n-1}(x - a_j)\,\mathrm{d}x$$

where

$$b_1 < b_2 < \cdots < b_{n-1}$$
$$a_1 < a_2 < \cdots < a_{n-1}.$$

Now we will study the system

$$\begin{cases} \dot{x} = (Ax + By + C)\partial_y V(x, y) \equiv P(x, y) \\ \dot{y} = -(Ax + By + C)\partial_x V(x, y) + \epsilon(V(x, y) - c_0) \equiv Q(x, y) \end{cases} \tag{4.1}$$

for which the curve

$$V(x, y) = c_0 \tag{4.2}$$

is its invariant curve, where A, B, C, ϵ, c_0 are certain real parameters.

Proposition 4.1. *There are values of the parameters A, B, C, ϵ, c_0 and b_j, a_j, $j = 1, 2, \ldots, n-1$ for which system (4.1) admits $\frac{1}{2}(n-1)(n-2)+1$ algebraic limit cycles when n is even and $\frac{1}{2}(n-1)(n-2)$ when n is odd, i.e.*

$$A(n, 1) \geqslant \begin{cases} \frac{1}{2}(n-1)(n-2)+1 & \text{if} \quad n \text{ is even} \\ \frac{1}{2}(n-1)(n-2) & \text{if} \quad n \text{ is odd}. \end{cases}$$

In fact there are values of the parameters a_j, b_j, c_0, $j = 1, 2, \ldots, n-1$ for which the curve (4.2) admits these numbers of ovals.

If we denote by (x_j, b_j) the point on the plane such that

$$-(Ax + Bb + C)\partial_x V(x, y) + \epsilon(V(x, y) - c_0)|_{x=x_j, y=b_j} = 0.$$

Then, by choosing the free parameters in such a way that the value of the Liapunov quantities

$$\begin{cases} \sigma(x, y) = \partial_x P(x, y) + \partial_y Q(x, y) \\ \triangle(x, y) = \partial_x P(x, y)\partial_y Q(x, y) - \partial_y P(x, y)\partial_x Q(x, y) \end{cases} \tag{4.3}$$

at some of the critical points which are in the interior of the ovals, we can see that the next inequality holds

$$\triangle(x_j, b_j) > 0 \qquad \sigma^2(x_j, b_j) - 4\triangle(x_j, b_j) < 0.$$

From these conditions we can construct the polynomial vector field of degree n with the indicated numbers of algebraic limit cycles.

Obviously, if $B\epsilon = 0$ then there is a first integral. In fact if $\epsilon = 0$ then V is an analytic first integral, if $B = 0$ then

$$F(x, y) = \frac{|V(x, y) - c_0|^A}{\left|x + \frac{C}{A}\right|^\epsilon}$$

is a Darboux first integral, where $A \neq 0$.

Example 4.1. The quadratic system

$$\begin{cases} \dot{x} = (Ax + By + C)y \\ \dot{y} = -(Ax + By + C)x + \epsilon(x^2 + y^2 - c_0) \end{cases}$$

admits one algebraic limit cycle for some values of the parameters.

4.2. A lower bound for $A(n, 2)$

Proposition 4.2. *Let us suppose that the straight line $x - a = 0$ does not intersect the ovals of the curve*

$$\epsilon x^n + G(x, y) = 0 \qquad n \geqslant 3.$$

Then there are values of the parameters $A, B, C, \beta, a, \epsilon$ for which system (3.2) admits $\frac{1}{2}(n-1)(n-2)+1$ algebraic limit cycles when n is odd and $\frac{1}{2}(n-1)(n-2)$ when n is even, i.e.,

$$A(n, 2) \geqslant \begin{cases} \frac{1}{2}(n-1)(n-2)+1 & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1)(n-2) & \text{if } n \text{ is odd.} \end{cases}$$

The case when $\epsilon = 0$ was studied in [Dol]. The authors of this paper proved that the polynomial vector field of degree n

$$\begin{cases} \dot{x} = (x-a)\partial_y G(x, y) \\ \dot{y} = -(x-a)\partial_x G(x, y) + (Ax + By + C)G(x, y) \qquad B \neq 0 \end{cases}$$

where $\deg G = n - 1$ admits $\frac{1}{2}(n-2)(n-3)$, or $\frac{1}{2}(n-2)(n-3)+1$ algebraic limit cycles if n is even or odd respectively.

4.3. An upper bound for $A(n, S)$

Proposition 4.3. *Let us suppose that the system (2.2) + (2.3) + (2.11) is not Darboux integrable, then*

$$\begin{cases} A(n, S) \leqslant \frac{1}{4}(n^2 - 3n + 4)(n - r)(n - r + 1) \\ 1 \leqslant r = \deg\{g_1, g_2\} \leqslant n - 2. \end{cases} \tag{4.4}$$

In fact, from proposition 3.1 we have that, if the polynomial system is not Darboux integrable, then the maximal number of the algebraic invariant curves is

$$2 \leqslant S \leqslant \tfrac{1}{2}(n-r)(n+1-r).$$

If we denote by K_j and m_j the number of ovals and the degree of the polynomial $g_j(x, y)$ then from the Harcnack theorem and proposition 3.3 we get that

$$K_j \leqslant \tfrac{1}{2}\left(m_j^2 - 3m_j + 4\right) \leqslant \tfrac{1}{2}(n^2 - 3n + 4)$$

hence,

$$A(n, S) \leqslant \sum_{j=1}^{S} K_j \leqslant \frac{1}{2}(n^2 - 3n + 4)S.$$

From here we easily deduce the following consequence.

Corollary 4.1. *For $r = n - 2$, we have that*

$$\tfrac{1}{2}(n-1)(n-2)+1 \geqslant A(n, 2) \geqslant \begin{cases} \frac{1}{2}(n-1)(n-2)+1 & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1)(n-2) & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 4.2. $A(n, 2) = \frac{1}{2}(n-1)(n-2) + 1 \quad$ *if n is even.*

Conjecture.

$$A(n,2) = \tfrac{1}{2}(n-1)(n-2) + 1 \qquad \forall n \in \mathcal{N}$$

Corollary 4.3.

$$A(3,2) = 2.$$

In fact, from corollary 4.1 we have that $1 \leqslant A(3,2) \leqslant 2$.

On the other hand, the cubic system

$$\begin{cases} \dot{x} = (F_0(x,y) - F_a(x,y))y \\ \dot{y} = -(F_0(x,y) - F_a(x,y))x + aF_0(x,y) \end{cases}$$

admits two limit cycles

$$x^2 + y^2 = r^2$$
$$(x-a)^2 + y^2 = r^2 \qquad a > 2r$$

where $F_a(x,y) = (x+y-a)((x-a)^2 + y^2 - r^2)$.

4.4. A lower bound for $A(n, n-1)$

Now we will construct a polynomial vector field of degree n with $n-1$ invariant circumferences, therefore, we will show that

$$A(n, n-1) \geqslant n-1.$$

The aim of the next assertions is to study the particular case when all the invariant curves of (2.2) + (2.3) are

$$g_j(x,y) = \tfrac{1}{2}((x-a_j)^2 + y^2) - r_j^2 \qquad j = 1,2,\ldots,S \tag{4.5}$$

where $a_1 \neq a_2, a_3, \ldots, a_S$ and $r_1, r_2, \ldots, r_S$ are real constants.

Proposition 4.4 ([Sad2]). *In the case when*

$$\begin{cases} a_1 = a_{l+2} = \cdots = a_S \\ a_2 = a_3 = \cdots = a_{l+1} \end{cases}$$

the polynomials μ_j, $\quad j = 1,2,\ldots,S$, *admit the representation*

$$\begin{cases} \mu_j(x,y) = P_{k_1}(x,y) \displaystyle\prod_{k=1,l+2,k\neq j}^{S} \left(g_1 + r_1^2 - r_k^2\right) \qquad j = 1, l+2, \ldots, S \\ \mu_j(x,y) = P_{k_2}(x,y) \displaystyle\prod_{k=2,k\neq j}^{l+1} \left(g_2 + r_2^2 - r_k^2\right) \qquad j = 2, \ldots, l+1 \end{cases} \tag{4.6}$$

where $P_{k_j}(x,y)$, $j = 1,2$, *are arbitrary polynomials of degree* k_1, k_2*:*

$$k_1 \leqslant n - 2(S-l) \qquad k_2 \leqslant n - 2l. \tag{4.7}$$

In fact, from conditions (2.2) + (4.5) we obtain that

$$(a_1 - a_2)(\mu_1 g_1 - \mu_j g_j) = 0 \qquad j = l+2, \ldots, S$$
$$(a_1 - a_2)(\mu_2 g_2 - \mu_j g_j) = 0 \qquad j = 3, \ldots, l+1$$

hence, we have easily obtained (4.6).

Corollary 4.4. *The system (2.2) + (2.3) + (4.5) is Darboux integrable iff*

$$\begin{cases}\mu_2(x,y) = K_2\big(g_1 + r_1^2\big)\displaystyle\prod_{k=3}^{l+1}\big(g_2 + r_2^2 - r_k^2\big)\\ \mu_1(x,y) = K_1\big(g_2 + r_2^2\big)\displaystyle\prod_{k=l+2}^{S}\big(g_1 + r_1^2 - r_k^2\big)\end{cases}$$

where K_1, K_2 *are arbitrary polynomials:*

$$\deg \mu_j \leqslant n-2 \qquad j = 1, 2.$$

The Darboux first integral is the following [Sad2],

$$F(x,y) = \prod_{k=1}^{S} |g_j|^{\lambda_j}$$

where λ_j, $j = 1, 2, \ldots, S$ *are constant:*

$$\begin{cases}\lambda_j = \dfrac{K_2\big(r_j^2\big)}{\prod_{k=1,l+2}^{S}\big(r_k^2 - r_j^2\big)} & j = 1, l+2, \ldots, S\\ \lambda_j = \dfrac{K_1\big(r_j^2\big)}{\prod_{k=2,k\neq j}^{l+1}\big(r_k^2 - r_j^2\big)} & j = 2, \ldots, l+1.\end{cases} \tag{4.8}$$

Proof. From (3.1) + (4.6) it follows that

$$\begin{cases}\displaystyle\sum_{j=2}^{l+1}\frac{\lambda_j}{R_2^2 - r_j^2} = \frac{K_2(x,y)}{\prod_{j=2}^{l+1}\big(R_2^2 - r_j^2\big)}\\ \displaystyle\sum_{j=1,l+2}^{S}\frac{\lambda_j}{R_1^2 - r_j^2} = \frac{K_1(x,y)}{\prod_{j=1,l+2}^{S}\big(R_2^2 - r_j^2\big)}\end{cases}$$

where

$$K_j(x,y) = \frac{P_{k_j}}{(a_2 - a_1)\nu(x,y)y}$$

$$R_j^2 = (x - a_j)^2 + y^2 \qquad j = 1, 2.$$

Hence, we easily deduced (4.8). □

Let us study the case when the circumferences are centred at the points (0, 0) and $(a, 0)$.

Corollary 4.5. *Let us suppose that*

$$\begin{cases}n = 2l + 1\\ S = n - 1 = 2l\\ a_1 = a_{l+2} = a_{l+3} = \cdots = a_{2l} = 0\\ a_2 = a_3 = \cdots = a_{l+1} = a\end{cases} \tag{4.9}$$

then polynomials $\mu_1, \mu_2, \ldots, \mu_{2l}$ *are such that*

$$\mu_j(x, y) = (\alpha x + \beta y + \gamma_2) \prod_{k=2, k\neq j}^{l+1} g_k(x, y)$$

$$\mu_{j+l}(x, y) = (\alpha x + \beta y + \gamma_1) \prod_{k=l+2, k\neq j+l}^{2l+1} g_k(x, y)$$

$$\mu_{2l+1} \equiv \mu_1 \qquad g_{2l+1} \equiv g_1 \qquad j = 2, \dots, l+1$$

where $\alpha, \beta, \gamma_1, \gamma_2$ *are arbitrary constants.*

The proof follows from (4.6) + (4.7) by taking into consideration the fact that the polynomials P_{k_j}, $j = 1, 2$ are such that

$$k_j \leqslant 1 \qquad j = 1, 2$$

i.e.,

$$P_{k_j} = \alpha_j x + \beta_j y + \gamma_j \qquad j = 1, 2.$$

To construct the polynomial vector field of degree n with the partial integrals (4.5) + (4.8) it is necessary that the constants $\alpha_1, \alpha_2 \beta_1, \beta_2$ satisfy the following conditions:

$$\alpha_1 = \alpha_2 \equiv \alpha \qquad \beta_1 = \beta_2 \equiv \beta.$$

Under these conditions we obtain that the polynomial vector field of degree n is the following:

$$\begin{cases} \dot{x} = 2y \left((\alpha x + \beta y + \gamma_1) g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (\alpha x + \beta y + \gamma_2) \prod_{j=2}^{l+1} g_j \right) \\ \dot{y} = -2x \left((\alpha x + \beta y + \gamma_1) g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (\alpha x + \beta y + \gamma_2) \prod_{j=2}^{l+1} g_j \right) \\ \qquad + 2(\alpha x + \beta y + \gamma_1) g_1 \displaystyle\prod_{j=l+2}^{2l} g_j. \end{cases}$$

The constants $\alpha, \beta, \gamma_1, \gamma_2$ are determined in such a way that

(1) the straight lines $\alpha x + \beta y + \gamma_j = 0$, $j = 1, 2$ do not intersect the circumferences,
(2) all the critical points have the ordinate zero and do not lie on the circumferences,
(3) the vector field has a finite number of critical points.

Hence $\gamma_1 \neq \gamma_2$.

We will study the following two cases:

(1) $\gamma_1 = 0 \qquad \gamma_2 = -a\alpha$

(2) $\alpha = \frac{1}{2} \qquad \beta = 0 \qquad \gamma_1 = -\gamma_2 = \frac{1}{2}\gamma > r_{2l}$

where r_{2l} is the largest radius.

In the first case the above vector field takes the form

$$\begin{cases} \dot{x} = 2y \left((\alpha x + \beta y) g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (\alpha(x - a) + \beta y) \prod_{j=2}^{l+1} g_j \right) \\ \dot{y} = -2 \left((x - a)(\alpha x + \beta y) g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (\alpha(x - a) + \beta y) x \prod_{j=2}^{l+1} g_j \right). \end{cases}$$

We will study the subcase when

$$\begin{cases}\alpha = \beta = \frac{1}{2} \\ r_1 = r_2 \qquad r_j = r_{l+(j-1)} \qquad j = 3, \ldots, l+1.\end{cases}$$

Designating by $F_a(x, y)$, $F_0(x, y)$ the following polynomials,

$$\begin{cases}F_a(x, y) = (x + y - a)\displaystyle\prod_{j=2}^{l+1}\left((x-a)^2 + y^2 - r_j^2\right) \qquad l \geqslant 1 \\ F_0(x, y) = F_a(x, y)|_{a=0}\end{cases}$$

we can deduce that the above vector field takes the form

$$\begin{cases}\dot{x} = (F_0(x, y) - F_a(x, y))y = P(x, y) \\ \dot{y} = -(F_0(x, y) - F_a(x, y))x + aF_0(x, y) = Q(x, y).\end{cases}$$

This system has the following properties.

(1) It has only three critical points in the finite plane R^2,

$$(0, 0) \qquad \left(\frac{a}{2}, 0\right) \qquad (a, 0).$$

(2) The Liapunov quantities (4.3) for the system are

(i)

$$\begin{cases}\sigma(0, 0) = \sigma(a, 0) \\ \Delta(0, 0) = \Delta(a, 0).\end{cases}$$

(ii)

$$\begin{cases}\sigma(0, 0) = (-1)^l a \displaystyle\prod_{j=1}^{l} r_j^2 \\ \Delta(0, 0) = a^2 \displaystyle\prod_{j=1}^{l} l\left(a^2 - r_j^2\right)\left(\prod_{j=1}^{l}\left(a^2 - r_j^2\right) - (-1)^l \prod_{j=1}^{l} r_j^2\right) \\ \sigma^2(0, 0) - 4\Delta(0, 0) = a^2\left(\left(2\displaystyle\prod_{j=1}^{l}\left(a^2 - r_j^2\right) - (-1)^l\prod_{j=1}^{l} r_j^2\right)^2 - 8\left(\prod_{j=1}^{l}\left(a^2 - r_j^2\right)\right)^2\right)\end{cases}$$

$$\begin{cases}\sigma\left(\dfrac{a}{2}, 0\right) = a\displaystyle\prod_{j=1}^{l}\left(\left(\frac{a}{2}\right)^2 - r_j^2\right) \\ \Delta\left(\dfrac{a}{2}, 0\right) = -\dfrac{a^4}{2}\displaystyle\prod_{j=1}^{l}\left(\left(\frac{a}{2}\right)^2 - r_j^2\right)\sum_{l=1}^{l}\prod_{j=1, j\neq l}^{l}\left(\left(\frac{a}{2}\right)^2 - r_j^2\right).\end{cases}$$

The circumferences are not intersected if

$$r_j < a/2 \qquad j = 1, \ldots, l$$

so

$$\Delta\left(\frac{a}{2}, 0\right) < 0$$

and, as a consequence the critical point $\left(\frac{a}{2}, 0\right)$ is a saddle.

It is evident that the other critical points are the stability or non-stability foci depending on whether k is odd or even.

The second case can be analysed analogously. The constants can be chosen as follows,

$$\alpha = \tfrac{1}{2} \qquad \beta = 0 \qquad \gamma_1 = -\gamma_2 = \tfrac{1}{2}\gamma > r_{2l}$$

where r_{2l} is the greater ratio.

Under these conditions the constructed vector field takes the form

$$\begin{cases} \dot{x} = y\left((x+\gamma)g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (x-\gamma)\prod_{j=2}^{l+1} g_j\right) \\ \dot{y} = -x\left((x+\gamma)g_1 \displaystyle\prod_{j=l+2}^{2l} g_j - (x-\gamma)\prod_{j=2}^{l+1} g_j\right) + (x+\gamma)g_1 \displaystyle\prod_{j=l+2}^{2l} g_j. \end{cases}$$

Hence, $S = n-1$ given circumferences are isolated periodical solutions of the polynomial vector field, i.e., are algebraic limit cycles.

From the above it follows that

$$A(n, n-1) \geqslant n-1.$$

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